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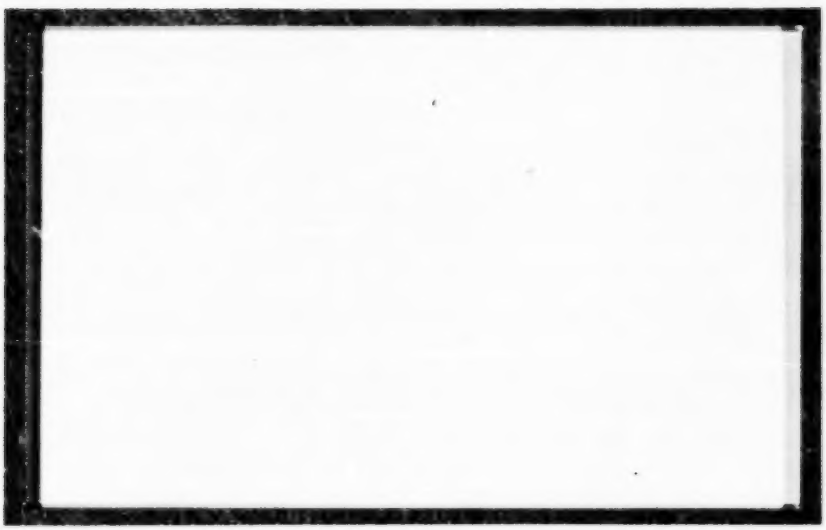

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VEHICLE ROUTING WITH FULL LOADS

by
Jacques Desrosiers*, Gilbert Laporte*
and François Soumis**

* GERAD, École des Hautes Études Commerciales, Montréal 5255 avenue Decelles, Montréal, CANADA H3T 1V6

** GERAD and École Polytechnique de Montréal, C.P. 6079, Station A, Montréal CANADA H3C3A7

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7. Abstract

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ABSTRACT

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RESUME

On considère dans cet article un problème de construction de tournées avec chargements entiers et des limites imposées sur la longueur des routes. On peut formuler ce problème comme un problème du voyageur de commerce asymétrique, assujetti à deux types de restrictions. Lorsque ces restrictions ne sont pas trop contraignantes, le problème peut être résolu de façon optimale pour des tailles relativement élevées.

1. INTRODUCTION

In a recent paper [1], Skitt and Levary described a column generation algorithm for a routing problem with full loads and maximum route length constraints. The purpose of this note is to provide an alternative formulation for this problem. When the length constraints are not too stringent, the new formulation lends itself to an efficient treatment since the problem is then almost identical to the asymmetrical vehicle routing problem (VRP) under distance restrictions, recently solved optimally for relatively large sizes [3].

The problem treated by Skitt and Levary can be represented by a graph $G = (M, A, C)$ where $M = \{1, \ldots, m\}$ is a set of nodes or cities, $A = \{(i,j): i \neq j, i,j \in M\}$ is a set of arcs and $C = (c_{ij})$ is the matrix of shortest distances or travel times between all pairs of nodes. In [5], C is symmetrical, but this need not always be the case. Nodes $s \in \{1, \ldots, r: r < m\}$ are used as *depots*. A fleet of w_s identical vehicles is based at depot s. With G is associated a demand matrix $\Delta = (\delta_{ij})$: δ_{ij} represents the number of *full vehicle loads* which must be shipped from node i to node j. In general, Δ is asymmetrical and relatively sparse.

The problem consists of designing distribution routes in such a way that

(i) all loads are moved from their origin to their destination;

(ii) every vehicle starts and ends its journey at its depot;

(iii) the length of a route (expressed in distance units, say) does not exceed a prespecified limit T;

(iv) the total distance travelled is minimized.

Since every arc (i,j) will have to be used for at least δ_{ij} trips in the optimal solution, the total distance associated with these compulsory trips can be considered as a constant. The problem really consists of minimizing the total distance travelled by empty vehicles. As in [2],

we will therefore replace every arc (i,j) corresponding to a trip ($\delta_{ij} > 0$) by δ_{ij} *trip nodes* and create *intertrip arcs* between these nodes. The problem now consists of minimizing the total distance travelled on intertrip arcs.

2. TRANSFORMED GRAPH

The new formulation is obtained by first replacing G by an extended graph H = (N, B, D), defined as follows.

Nodes of H

N is the union of N_1, a set of *vehicle nodes* and N_2, a set of *trip nodes* where

(1) $N_1 = \{h_{st}: s=1, \ldots, r; \quad t=1, \ldots, w_s+1\}$

$N_2 = \{h_{ij\mu}: i,j \in M; \quad \delta_{ij} > 0; \quad \mu = 1, \ldots, \delta_{ij}\}$

The cardinality of N is therefore equal to

(2) $$n = r + \sum_{s=1}^{r} w_s + \sum_{i,j \in M} \delta_{ij}$$

To every trip node $K = h_{ij\mu}$ is associated a distance $d_K = c_{ij}$.

Arcs of H

(3) $B = \{(I,J): I \neq J, \ I,J \in N\}$

Distance matrix of H

(4) $D = (d_{IJ})$

where the d_{IJ}'s are associated with the arcs of B and are defined in the following way.

(i) Intertrip arcs: $I,J \in N_2$. Let $I = h_{ij\mu}$ and $J = h_{k\ell\nu}$. Then $d_{IJ} = c_{jk}$.

(ii) Arcs associated with vehicles leaving a depot: $I \in N_1$, $J \in N_2$. Let $I = h_{st}$ and $J = h_{ij\mu}$. Then $d_{IJ} = c_{si}$.

(iii) Return arcs to a depot: $I \in N_2$, $J \in N_1$. Let $I = h_{ij\mu}$ and $J = h_{st}$. Then $d_{IJ} = c_{js}$.

(iv) Arcs associated with the use of a vehicle: $I,J \in N_1$. Let $I = h_{st}$ and $J = h_{uv}$. Then

$$(5) \quad d_{IJ} = \begin{cases} \lambda_s & \text{if } s = u \text{ and } v = t+1 \\ -\infty & \text{if } s < r,\ u = s+1,\ t = w_s+1,\ v = 1 \\ & \text{or } s = r,\ u = 1,\ t = w_r+1,\ v = 1 \\ \infty & \text{otherwise} \end{cases}$$

where, as in [4], $-\lambda_s$ can be interpreted as the cost of using a vehicle at depot s. Thus,

(*) if $\lambda_s = \infty$, all w_s vehicles based at depot s will be used;

(**) if $\lambda_s = -\infty$, the minimum possible number of vehicles based at depot s will be used;

(***) if $\lambda_s = 0$, the number of vehicles used at depot s is unspecified: it may take any value between the smallest possible value and w_s.

The passage from G to H can be illustrated by this example: $M = \{1, 2, 3, 4, 5, 6\}$, $r = 2$, $w_1 = w_2 = 1$, $\delta_{1,3} = \delta_{3,4} = 1$; $\delta_{2,5} = \delta_{4,6} = 2$; $\delta_{ij} = 0$ otherwise. Then H is defined by the following tableau.

		N_1 : vehicle nodes				N_2 : trip nodes					
		$h_{1,1}$	$h_{1,2}$	$h_{2,1}$	$h_{2,2}$	$h_{1,3,1}$	$h_{2,5,1}$	$h_{2,5,2}$	$h_{3,4,1}$	$h_{4,6,1}$	$h_{4,6,2}$
N_1	$h_{1,1}$	-	λ_1	∞	∞	0	$c_{1,2}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,4}$
	$h_{1,2}$	∞	-	$-\infty$	∞	0	$c_{1,2}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,4}$
	$h_{2,1}$	∞	∞	-	λ_2	$c_{2,1}$	0	0	$c_{2,3}$	$c_{2,4}$	$c_{2,4}$
	$h_{2,2}$	$-\infty$	∞	∞	-	$c_{2,1}$	0	0	$c_{2,3}$	$c_{2,4}$	$c_{2,4}$
N_2	$h_{1,3,1}$	$c_{3,1}$	$c_{3,1}$	$c_{3,2}$	$c_{3,2}$	-	$c_{3,2}$	$c_{3,2}$	0	$c_{3,4}$	$c_{3,4}$
	$h_{2,5,1}$	$c_{5,1}$	$c_{5,1}$	$c_{5,2}$	$c_{5,2}$	$c_{5,1}$	-	$c_{5,2}$	$c_{5,3}$	$c_{5,4}$	$c_{5,4}$
	$h_{2,5,2}$	$c_{5,1}$	$c_{5,1}$	$c_{5,2}$	$c_{5,2}$	$c_{5,1}$	$c_{5,2}$	-	$c_{5,3}$	$c_{5,4}$	$c_{5,4}$
	$h_{3,4,1}$	$c_{4,1}$	$c_{4,1}$	$c_{4,2}$	$c_{4,2}$	$c_{4,1}$	$c_{4,2}$	$c_{4,2}$	-	0	0
	$h_{4,6,1}$	$c_{6,1}$	$c_{6,1}$	$c_{6,2}$	$c_{6,2}$	$c_{6,1}$	$c_{6,2}$	$c_{6,2}$	$c_{6,3}$	-	$c_{6,4}$
	$h_{4,6,2}$	$c_{6,1}$	$c_{6,1}$	$c_{6,2}$	$c_{6,2}$	$c_{6,1}$	$c_{6,2}$	$c_{6,2}$	$c_{6,3}$	$c_{6,4}$	-

3. FORMULATION

The problem now consists of determining a minimum length Hamiltonian circuit on H, subject to two types of constraints corresponding to the following requirements in the original problem:

(i) every vehicle starts and ends its journey at the same depot;

(ii) the length of any vehicle route never exceeds T.

In order to derive analytical expressions corresponding to these two constraints, first consider the solution associated with the unrestricted problem. This solution corresponds to a permutation $(I_1, \ldots, I_n)$ of all nodes of N which can be broken into $n_1 = r + \sum_{i=1}^{r} w_s$ paths P_k of the form $(I_{\alpha(k)}, \ldots, I_{\beta(k)})$ where

(i) $I_{\alpha(k)}, I_{\beta(k)} \in N_1$ $\quad (k=1, \ldots, n_1)$

(ii) $I_{\alpha(k)+1}, \ldots, I_{\beta(k)-1} \in N_2$ $\quad (\beta(k) > \alpha(k)+1;\ k=1, \ldots, n_1)$

(iii) $I_{\alpha(k)} = I_{\beta(k-1)}$ $\quad (k=2, \ldots, n_1)$

$I_{\alpha(1)} = I_{\beta(n_1)}$

Consider only paths P_k for which $\beta(k) > \alpha(k)+1$. Path P_k is illegal if

(i) it connects two different depots:

$$I_{\alpha(k)} = h_{st},\ I_{\beta(k)} = h_{uv} \text{ and } s \neq u \text{ or,}$$

(ii) its length $L(P_k)$ exceeds T:

$$L(P_k) = \sum_{K=I_{\alpha(k)}}^{I_{\beta(k)-1}} d_{K,K+1} + \sum_{K=I_{\alpha(k)+1}}^{I_{\beta(k)-1}} d_K > T$$

The first summation corresponds to the total intertrip distance of P_k, while the second summation represents the total distance associated with the compulsory trips of P_k.

Any illegal path P_k can be eliminated by forcing at least one of its arcs to 0, i.e. by imposing that the number of its arcs be at most equal to $\beta(k) - \alpha(k) - 2$.

The original problem can now be formulated as a constrained travelling salesman problem (TSP) on H. Let x_{IJ} be a 0-1 variable indica-

ting whether arc (I,J) enters the solution ($x_{IJ} = 1$) or not ($x_{IJ} = 0$). The problem is then

(P1) minimize $\sum_{I,J} d_{IJ} x_{IJ}$

subject to

(6) $\sum_{I \neq J} x_{IJ} = 1$ $(J=1, \ldots, n)$

(7) $\sum_{J \neq I} x_{IJ} = 1$ $(I=1, \ldots, n)$

(8) $\sum_{I,J \in S} x_{IJ} \leq |S| - 1$ $(S \subset N;\ |S| \geq 2)$

(9) $\sum_{I,J \in Q(k)} x_{IJ} \leq \beta(k) - \alpha(k) - 2$ (where $P_k = (I_{\alpha(k)}, \ldots, I_{\beta(k)})$ is illegal and Q(k) is the node set of P_k)

(10) $x_{IJ} = 0, 1$ $(I,J \in N)$

In (P1), constraints (6), (7) and (10) are those of the assignment problem. Constraints (8) are the classical TSP subtour elimination constraints: they prohibit the formation of tours over proper subsets S of N. The justification and derivation of constraints (9) have already been provided.

4. ALGORITHM

The problem defined by (P1) is a direct extension of the asymmetrical TSP for which efficient algorithms have been developed [1] and of the distance constrained TSP (denoted by DVRP) recently treated by Laporte et al. [3]. The only constraints present in (P1) but not in the

DVRP are those forbidding routes between two different depots. But these constraints are structurally similar to those probihiting the formation of routes of length exceeding T and can therefore be treated in the same fashion.

Results obtained by Laporte et al. indicate that DVRP's involving up to 400 nodes can be solved to optimality within a reasonable time provided the length constraint is not too restrictive. As the constraints becomes more binding, only smaller size problems can be solved optimally. In the example solved by Skitt and Levary, there are 45 trip nodes and the number of vehicles is not specified. Assuming there are 5 vehicles based at each of the 2 depots (the solution presented in [5] uses only 4 vehicles), the total number of nodes in H would then be 57, which is well within the capability of the algorithm described in [3].

The algorithm developed for the DVRP is a direct extension of the Carpaneto and Toth algorithm for the asymmetrical TSP [1]. For the sake of brevity, we will not provide a full description of the Carpaneto and Toth algorithm, but only outline its structure. This algorithm solves the TSP by means of a branch and bound tree. Subtour elimination constraints (8) are initially relaxed and only introduced as they are found to be violated. The problem solved at every node g of the search tree is a modified assignment problem (assignments on the main diagonal are prohibited) constrained by a set E_g of arcs fixed at 0 and a set F_g of arcs fixed at 1. Branching is always done from the illegal subtour involving the least number of nodes not already fixed at 1. A breadth first branching procedure is used. The process ends when all branches of the search tree have been explored according to the usual branch and bound rules.

Violated constraints (9) can be treated in the same way as illegal subtours. Moreover, in order to benefit from the particular structure of the problem at hand, the following features can be incorporated into the algorithm.

(i) In G, any arc (i,j) for which $c_{si} + c_{ij} + c_{js} > T$ for all $s = r$ can be eliminated by setting $c_{ij} = \infty$.

(ii) If at a given node of the search tree the value of a known lower bound on the optimum exceeds $T \times \sum_{s=1}^{r} w_s$, the node has no feasible descendant and the branch can be fathomed.

(iii) At any node g of the search tree, consider all paths $P_k = (I_{\alpha(k)}, \ldots, I_{\beta(k)})$ where $I_{\alpha(k)+1}, \ldots, I_{\beta(k)-1} \in N_2$ and for which all arcs $(I_{\alpha(k)}, I_{\alpha(k)+1}), \ldots, (I_{\beta(k)-1}, I_{\beta(k)})$ are elements of F_g (i.e. they are currently fixed at 1). As in section 3, let $L(P_k)$ be the length of P_k. Then arc (K,L) can be forced to 0 if

(*) $L=I_{\alpha(k)} \in N_2$ and $d_{KL} + L(P_k) > T$ or,

(**) $K=I_{\beta(k)} \in N_2$ and $d_{KL} + L(P_k) > T$ or,

(***) $L=I_{\alpha(k)} \in N_2$; $K=h_{st} \in N_1$, $I_{\beta(k)}=h_{uv} \in N_1$ and $s \neq u$ or,

(****) $K=I_{\beta(k)} \in N_2$; $L=h_{st} \in N_1$, $I_{\alpha(k)}=h_{uv} \in N_1$ and $s \neq u$.

(iv) Similarly, consider two paths $P_k = (I_{\alpha(k)}, \ldots, I_{\beta(k)})$ and $P_\ell = (I_{\alpha(\ell)}, \ldots, I_{\beta(\ell)})$ defined as in (iii). Then arc (K,L) can be forced to 0 if

(*) $K=I_{\beta(k)} \in N_2$, $L=I_{\alpha(\ell)} \in N_2$ and $d_{KL} + L(P_k) + L(P_\ell) > T$ or,

(**) $K=I_{\beta(k)} \in N_2$, $L=I_{\alpha(\ell)} \in N_2$; $I_{\alpha(k)}=h_{st} \in N_1$, $I_{\beta(\ell)}=h_{uv} \in N_1$ and $s \neq u$.

(v) Consider a node g of the search tree and $(I,J) \in E_g$ where $I \in N_1$ ($I = h_{st}$, say) and $J \in N_2$. Then all arcs (I',J) for which $I' = h_{st'}$ can be included in E_g. The same rule applies *mutatis mutandis* if $I \in N_2$ and $J \in N_1$.

5. CONCLUSION

The vehicle routing problem with full loads described by Skitt and Levary can be transformed into a TSP with additional restrictions. Provided the problem size is not too large and the maximum length constraints are not too binding, the problem can be solved efficiently by means of a direct extension of the Carpaneto and Toth branch and bound algorithm for the TSP.

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